ON NONLINEAR TENSOR FUNCTIONS IN THE THEORY OF RELATIVITY

(O NELINEINYKH TENSORNYKH FUNKTSIIAKH) V TEORII OTNOSITEL'NOSTI)

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We shall consider the construction of a general form of nonlinear tensor function invariant with respect to the group G of such orthogonal transformations of Cartesian coordinates defining the space tangent to the four-dimensional Riemannian manifold L at some point M, which leave the vector u^1 of unit length defined in M, at rest. Investigation of such tensor functions id of interest since the group G of the form indicated above, defines the properties of spatial symmetry of the medium in the relativistic mechanics of continuous media, provided that u^1 is taken as the 4-velocity vector (*).

Let a tensor A of rank r whose components are $A^{i_1 \ldots i_r}$, be given on the space-time manifold L. Let it also be, by definition, invariant with respect to the group G, which represents the group of orthogonal transformations of coordinates in the associated reference system (**). When acted upon by the elements of G, components of $A^{*\alpha_1 \ldots \alpha_r}$ will transform in the manner of components of a three-dimensional tensor of rank r, components $A^{*4\alpha_2 \ldots \alpha_r}$ $\ldots, A^{*\alpha_1 \ldots \alpha_{r-1} 4}$ like the components of a three-dimensional tensor of rank $r-1$, etc. The component $A^{*4 \ldots 4}$ is, under the transformations of G, a scalar. Results of [1] enable us to write the general form of three-dimensional tensors $A^{*\alpha_1 \ldots \alpha_r}$, $A^{*4\alpha_2 \ldots \alpha_r}, \ldots$ in the associated reference system. In other reference systems, components of the tensor A are obtained by coordinate transformations.

We shall illustrate the above argument using a tensor of rank 2 invariant with respect to a complete orthogonal group of transformations of coordinates of the associated reference system. From [1] it follows that

$$A^{*\alpha\beta} = k_1{}^* g^{*\alpha\beta}, \qquad A^{*\alpha 4} = A^{*4\alpha} = 0, \qquad A^{*44} = k_2{}^* \tag{1}$$

where g^{ij} are the components of the metric tensor of L, while $k_1{}^*$ and $k_2{}^*$ are scalars with respect to the complete orthogonal group of transformations of coordinates $x^{*\alpha}$. It is easy to find a tensor whose components in the associated reference system coincide with those of (1)

$$A^{ij} = k_1 \gamma^{ij} + k_2 u^i u^j \tag{2}$$

*) Latin letter indices $i, j, k, \ldots$ assume the values 1, 2, 3 and 4; Greek letter indices $\alpha, \beta, \gamma, \ldots$ correspond to spatial coordinates and assume the values 1, 2 and 3.
**) Magnitudes measured in the associated reference system (Cartesian reference system in the space tangent to L with one of its coordinate axes coinciding with the direction of u^1), will be denoted by the symbol *.

Here u^i is the 4-velocity vector, $\gamma^{ij} = g^{ij} - u^i u^j$ in the metric with the sign convention $(---+)$, $\gamma^{ij} = g^{ij} + u^i u^j$ in the metric with the sign convention $(+++-)$, while k_1 and k_2 are four-dimensional scalars, numerically equal to k_1^* and k_2^*. Since the tensors whose components coincide in any reference system are equal, Formula (2) yields the required result in any coordinate system.

Below we give the general form of invariant tensors of the first, second, third and fourth rank, for each of the seven types of orientations.

1) Class $\infty/\infty\cdot m$ (centrally isotropic case) (*). The group consists of all possible rotations and inversions. Generating elements of the group are: the intersecting axes of infinite order and the mirror plane of symmetry

$$A^i = ku^i, \quad A^{ij} = k_1\gamma^{ij} + k_2 u^i u^j$$

$$A^{ijk} = k_1\gamma^{ij}u^k + k_2\gamma^{ik}u^j + k_3\gamma^{jk}u^i + k_4 u^i u^j u^k$$

$$A^{ijkl} = k_1\gamma^{ij}\gamma^{kl} + k_2\gamma^{ik}\gamma^{jl} + k_3\gamma^{il}\gamma^{jk} + k_4\gamma^{ij}u^k u^l + k_5\gamma^{jk}u^i u^l + k_6\gamma^{kl}u^i u^j +$$
$$+ k_7\gamma^{il}u^j u^k + k_8\gamma^{ik}u^j u^l + k_9\gamma^{jl}u^i u^k + k_{10}u^i u^j u^k u^l$$

2) Class ∞/∞ (gyrotropic case). The group consists of all possible rotations with the determinant equal to $+1$. Generating elements of the group are the intersecting symmetry axes of infinite order

$$A^i = ku^i, \qquad A^{ij} = k_1\gamma^{ij} + k_2 u^i u^j$$

$$A^{ijk} = k_1\gamma^{ij}u^k + k_2\gamma^{ik}u^j + k_3\gamma^{jk}u^i + k_4 E^{ijkl}u_l + k_5 u^i u^j u^k$$

$$A^{ijkl} = A^{ijkl}(\infty/\infty\cdot m) + k_{11}E^{jklm}u_m u^i + k_{12}E^{iklm}u_m u^j + k_{13}E^{ijlm}u_m u^k + k_{14}E^{ijkm}u_m u^l$$

Here E^{ijkl} denotes a tensor antisymmetric in all its indices and $E^{1234} = \sqrt{g}$, where $g = \det\|g_{ij}\|$.

3) Class $m\cdot\infty : m$ (cylindrical symmetry). Elements of this group transform the cylinder into itself. Generating elements of this group are: the axis of infinite order together with the vertical and horizontal mirror symmetry planes

$$A^i = ku^i, \qquad A^{ij} = k_1\gamma^{ij} + k_2 B^{ij} + k_3 u^i u^j$$

$$A^{ijk} = A^{ijk}(\infty/\infty\cdot m) + k_5 B^{ij}u^k + k_6 B^{ik}u^j + k_7 B^{jk}u^i$$

$$A^{ijkl} = A^{ijkl}(\infty/\infty\cdot m) + k_{11}\gamma^{ij}B^{kl} + k_{12}\gamma^{ik}B^{jl} + k_{13}\gamma^{il}B^{jk} + k_{14}\gamma^{kl}B^{ij} + k_{15}\gamma^{jl}B^{ik} +$$
$$+ k_{16}\gamma^{jk}B^{il} + k_{17}B^{ij}B^{kl} + k_{18}B^{kl}u^i u^j + k_{19}B^{jl}u^i u^k + k_{20}B^{il}u^j u^k + k_{21}B^{jk}u^i u^l +$$
$$+ k_{22}B^{ik}u^j u^l + k_{23}B^{ij}u^k u^l$$

Here the tensors B^{ij} is defined by the following condition: its component matrix can, by means of spatial transformations, in the associated reference system, be reduced to the form $\|B^{*ij}\|$ $(i, j = 1, 2, 3, 4)$, in which the only component different from zero, will be B^{*33}.

4) Class $\infty : 2$ (gyrocylindrical symmetry). Elements of this group transform the cylinder into itself, with the determinant equal to $+1$. Generating elements of the group are: the axis of infinite order and a second order axis perpendicular to it

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*) Group notation used here is due to A. V. Shubnikov.

$$A^i = ku^i, \qquad A^{ij} = k_1\gamma^{ij} + k_2B^{ij} + k_3u^iu^j$$

$$A^{ijk} = A^{ijk}(\infty / \infty \cdot m) + k_5B^{ij}u^k + k_6B^{ik}u^j + k_7B^{jk}u^i + k_8E^{ijkl}u_l + k_9E^{ijlm}B^k{}_lu_m + \\ + k_{10}E^{jklm}B^i{}_lu_m$$

$$A^{ijkl} = A^{ijkl}(m \cdot \infty : m) + k_{24}E^{ijkm}u_mu^l + k_{25}E^{ijlm}u_mu^k + k_{26}E^{iklm}u_mu^j + \\ + k_{27}E^{jklm}u_mu^i + k_{28}E^{jksm}B^i{}_su_mu^l + k_{29}E^{ijsm}B^k{}_su_mu^l + k_{30}E^{klsm}B^j{}_su_mu^i + \\ + k_{31}E^{jksm}B^l{}_su_mu^i + k_{32}E^{klsm}B^i{}_su_mu^j + k_{33}E^{iksm}B^l{}_su_mu^j + k_{34}E^{jlsm}B^i{}_su_mu^k + \\ + k_{35}E^{ijsm}B^l{}_su_mu^k$$

5) Class $\infty : m$. Elements of this group transform a cylinder into itself, while preserving the basic orientation. Generating elements of the group are : the axis of infinite order and a mirror symmetry plane perpendicular to it

$$A^i = ku^i, \qquad A^{ij} = k_1\gamma^{ij} + k_2B^{ij} + k_3\Omega^{ij} + k_4u^iu^j$$

$$A^{ijk} = A^{ijk}(\infty / \infty \cdot m) + k_5B^{ij}u^k + k_6B^{ik}u^j + k_7B^{jk}u^i + k_8\Omega^{ij}u^k + k_9\Omega^{ik}u^j + k_{10}\Omega^{jk}u^i$$

$$A^{ijkl} = A^{ijkl}(m \cdot \infty : m) + k_{24}\gamma^{ij}\Omega^{kl} + k_{25}\gamma^{ik}\Omega^{jl} + k_{26}\gamma^{il}\Omega^{jk} + k_{27}\gamma^{kl}\Omega^{ij} + k_{28}\gamma^{jl}\Omega^{ik} + \\ + k_{29}\gamma^{jk}\Omega^{il} + k_{30}B^{ij}\Omega^{kl} + k_{31}B^{ik}\Omega^{jl} + k_{32}B^{kl}\Omega^{ij} + k_{33}\Omega^{kl}u^iu^j + k_{34}\Omega^{jl}u^iu^k + \\ + k_{35}\Omega^{il}u^ju^k + k_{36}\Omega^{jk}u^in^l + k_{37}\Omega^{ik}u^ju^l + k_{38}\Omega^{ij}u^ku^l$$

where the tensors B^{ij} and Ω^{ij} are defined by the following condition: their component matrices can, by simultaneous spatial coordinate transformation in the assoçiated reference system, be reduced to the form $\|B^{*ij}\|$ and $\|\Omega^{*ij}\|(i, j = 1, 2, 3, 4)$ in which the only components different from zero will be $B^{*33} = 1$ and $Q^{*12} = -Q^{*21} = 1$.

6) Class $\infty \cdot m$ (conical symmetry). Elements of this group transform the cone into itself. Generating elements of the group are: axis of infinite order and the mirror symmetry plane passing through it

$$A^i = k_1b^i + k_2u^i$$

$$A^{ij} = k_1\gamma^{ij} + k_2b^ib^j + k_3b^iu^j + k_4b^ju^i + k_5u^iu^j$$

$$A^{ijk} = A^{ijk}(\infty / \infty \cdot m) + k_5\gamma^{ij}b^k + k_6\gamma^{ik}b^j + k_7\gamma^{kj}b^i + k_8b^ib^jb^k + k_9b^ib^ju^k + \\ + k_{10}b^ib^ku^j + k_{11}b^jb^ku^i + k_{12}b^iu^ju^k + k_{13}b^ku^iu^j + k_{14}b^ju^ku^i$$

$$A^{ijkl} = A^{ijkl}(\infty / \infty \cdot m) + k_{11}\gamma^{ij}b^kb^l + k_{12}\gamma^{ik}b^jb^l + k_{13}\gamma^{il}b^jb^k + k_{14}\gamma^{kl}b^ib^j + \\ + k_{15}\gamma^{jl}b^ib^k + k_{16}\gamma^{jk}b^ib^l + k_{17}\gamma^{ij}b^ku^l + k_{18}\gamma^{ik}b^ju^l + k_{19}\gamma^{jk}b^iu^l + k_{20}\gamma^{ij}b^lu^k + \\ + k_{21}\gamma^{il}u^kb^j + k_{22}\gamma^{jl}b^iu^k + k_{23}\gamma^{il}b^ku^j + k_{24}\gamma^{ik}b^lu^j + k_{25}\gamma^{kl}b^iu^j + k_{26}\gamma^{jk}b^lu^i + \\ + k_{27}\gamma^{jl}b^ku^i + k_{28}\gamma^{lk}b^ju^i + k_{29}b^ib^jb^kb^l + k_{30}b^ib^jb^ku^l + k_{31}b^jb^kb^lu^i + k_{32}b^kb^lb^iu^j + \\ + k_{33}b^lb^ib^ju^k + k_{34}b^ib^ju^ku^l + k_{35}b^jb^ku^lu^i + k_{36}b^kb^lu^iu^j + k_{37}b^lb^iu^ju^k + k_{38}b^jb^lu^iu^k + \\ + k_{39}b^ib^ku^ju^l + k_{40}b^iu^ju^ku^l + k_{41}b^ju^iu^ku^l + k_{42}b^ku^iu^ju^l + k_{43}b^lu^iu^ju^k$$

Here the vector b^i can, in the associated reference system, be brought by means of spatial transformation, to the form $b^{*i} = (0, 0, 1, 0)$.

7) Class ∞ (gyroconical symmetry). Elements of this group transform the cone into itself, with the determinant equal to $+1$. Generating element of this group is the axis of infinite order

$$A^i = k_1 b^i + k_2 u^i$$

$$A^{ij} = k_1 \gamma^{ij} + k_2 \omega^{ij} + k_3 b^i b^j + k_4 b^i u^j + k_5 b^j u^i + k_6 u^i u^j$$

$$A^{ijk} = A^{ijk}(\infty \cdot m) + k_{15}\omega^{ij} b^k + k_{16}\omega^{ik} b^j + k_{17}\omega^{jk} b^i + k_{18}\omega^{ij} u^k + k_{19}\omega^{ik} u^j + k_{20}\omega^{jk} u^i$$

$$\begin{aligned} A^{ijkl} = A^{ijkl}(\infty \cdot m) &+ k_{44}\gamma^{ij}\omega^{kl} + k_{45}\gamma^{ik}\omega^{jl} + k_{46}\gamma^{il}\omega^{jk} + k_{47}\gamma^{kl}\omega^{ij} + \\ &+ k_{48}\gamma^{jl}\omega^{ik} + k_{49}\gamma^{jk}\omega^{il} + k_{50}\omega^{ij} b^k b^l + k_{51}\omega^{kl} b^i b^j + k_{52}\omega^{jl} b^i b^k + k_{53}\omega^{kl} b^j u^i + k_{54}\omega^{jl} b^k u^i + \\ &+ k_{55}\omega^{jk} u^i b^l + k_{56}\omega^{kl} b^i u^j + k_{57}\omega^{il} b^k u^j + k_{58}\omega^{ik} b^l u^j + k_{59}\omega^{jl} b^i u^k + k_{60}\omega^{il} b^j u^k + \\ &+ k_{61}\omega^{ij} b^l u^k + k_{62}\omega^{jk} b^i u^l + k_{63}\omega^{ik} b^j u^l + k_{64}\omega^{ij} b^k u^l + k_{65}\omega^{kl} u^i u^j + k_{66}\omega^{jl} u^i u^k + \\ &+ k_{67}\omega^{jk} u^i u^l + k_{68}\omega^{il} u^j u^k + k_{69}\omega^{ik} u^j u^l + k_{70}\omega^{ij} u^k u^l \end{aligned}$$

Here vector b^i is identical with that for the $(\infty \cdot m)$ group, while ω^{ij} are components of the tensor

$$\omega^{ij} = E^{ijkl} b_k u_l$$

In the above equations, coefficients k_1, k_2, ... can be any scalar functions of arbitrary magnitudes.

If the tensor A depends on tensor arguments A , A_2, ..., A_n and possesses the symmetry of one of the orientations, then the formulas given above represent a general form of such a relationship, while the coefficients k_1, k_2, ... represent the functions of invariants common to the system of tensors A_1, ..., A_n. If tensor A has a wider symmetry group, then constraints appear between the coefficients k_1, k_2, ...

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